Chaotic behavior of disordered nonlinear lattices

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Outline

- **Disordered lattices:**

  - ✓ **The quartic Klein-Gordon (KG) model**
  - ✓ **The disordered nonlinear Schrödinger equation (DNLS)**
  - ✓ **Different dynamical behaviors**

- **Chaotic behavior of the KG model**

  - ✓ **Lyapunov exponents**
  - ✓ **Deviation Vector Distributions**

- **Numerical methods**

  - ✓ **Symplecic Integrators**
 - ✓ **Tangent Map method**

- **Summary**

Interplay of disorder and nonlinearity

Waves in disordered media – Anderson localization [Anderson, Phys. Rev. (1958)]. **Experiments on BEC** [Billy et al., Nature (2008)]

Waves in nonlinear disordered media – localization or delocalization?

Theoretical and/or numerical studies [Shepelyansky, PRL (1993) – Molina, Phys. Rev. B (1998) – Pikovsky & Shepelyansky, PRL (2008) – Kopidakis et al., PRL (2008) – Flach et al., PRL (2009) – S. et al., PRE (2009) – Mulansky & Pikovsky, EPL (2010) – S. & Flach, PRE (2010) – Laptyeva et al., EPL (2010) – Mulansky et al., PRE & J.Stat.Phys. (2011) – Bodyfelt et al., PRE (2011) – Bodyfelt et al., IJBC (2011)]

Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)]
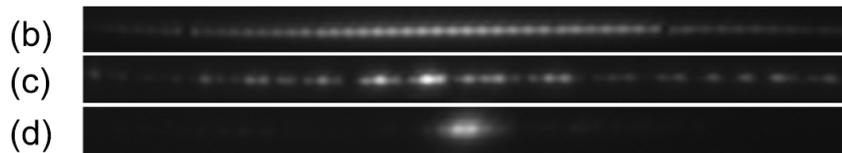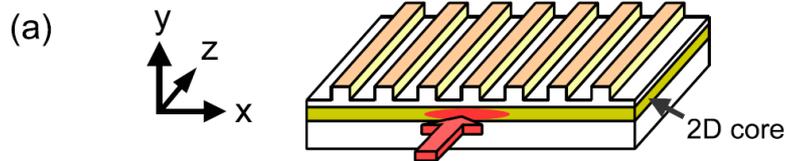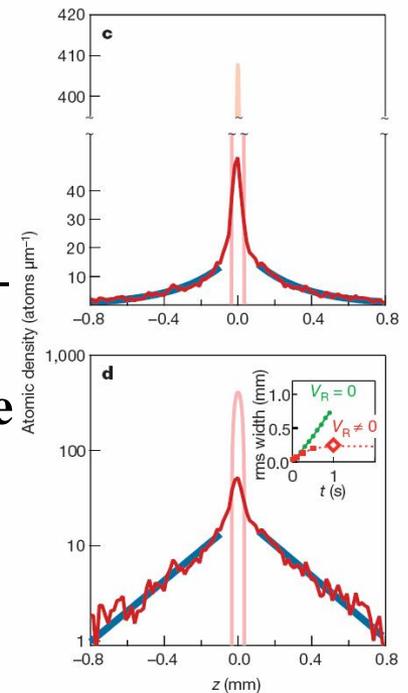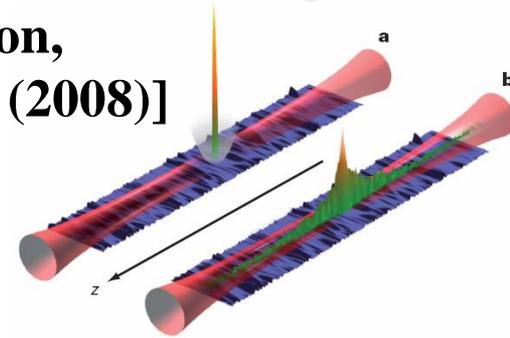
The Klein – Gordon (KG) model

$$H_K = \sum_{l=1}^{N} \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} \left(u_{l+1} - u_l \right)^2$$

with **fixed boundary conditions** $u_0 = p_0 = u_{N+1} = p_{N+1} = 0$. **Typically N=1000.**

Parameters: W and the **total energy E.** $\tilde{\varepsilon}_l$ **chosen uniformly from** $\left[\frac{1}{2}, \frac{3}{2} \right]$.

Linear case (neglecting the term $u_l^4/4$)

Ansatz: $u_l = A_l \exp(i\omega t)$. **Normal modes (NMs)** $A_{v,l}$ - **Eigenvalue problem:**

$$\lambda A_l = \varepsilon_l A_l - (A_{l+1} + A_{l-1}) \text{ with } \lambda = W\omega^2 - W - 2, \quad \varepsilon_l = W(\tilde{\varepsilon}_l - 1)$$

The discrete nonlinear Schrödinger (DNLS) equation

We also consider the system:

$$H_D = \sum_{l=1}^{N} \varepsilon_l |\psi_l|^2 + \frac{\beta}{2} |\psi_l|^4 - \left(\psi_{l+1} \psi_l^* + \psi_{l+1}^* \psi_l \right)$$

where ε_l **chosen uniformly from** $\left[-\frac{W}{2}, \frac{W}{2} \right]$ and β **is the nonlinear parameter.**

Conserved quantities: The energy and the norm $S = \sum_l |\psi_l|^2$ **of the wave packet.**

Distribution characterization

We consider normalized **energy distributions** in normal mode (NM) space

$$z_v \equiv \frac{E_v}{\sum_m E_m} \quad \text{with} \quad E_v = \frac{1}{2}\left(\dot{A}_v^2 + \omega_v^2 A_v^2\right) \quad , \text{ where } A_v \text{ is the amplitude}$$

of the vth NM (KG) or **norm distributions** (DNLS).

Second moment: $\quad m_2 = \sum_{v=1}^{N}\left(v - \bar{v}\right)^2 z_v \quad$ with $\quad \bar{v} = \sum_{v=1}^{N} v z_v$

Participation number: $\quad P = \dfrac{1}{\sum_{v=1}^{N} z_v^2}$

measures the number of stronger excited modes in z_v.

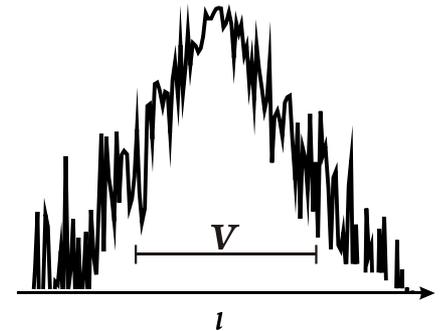Single mode $P=1$. Equipartition of energy $P=N$.

Scales

Linear case: $\omega_v^2 \in \left[\dfrac{1}{2}, \dfrac{3}{2} + \dfrac{4}{W}\right]$, width of the squared frequency spectrum:

$$\Delta_K = 1 + \dfrac{4}{W}$$

$$(\Delta_D = W + 4)$$

Localization volume of an eigenstate:

$$V \sim \dfrac{1}{\displaystyle\sum_{l=1}^{N} A_{v,l}^4}$$

Average spacing of squared eigenfrequencies of NMs within the range of a localization volume: $\quad d_K \approx \dfrac{\Delta_K}{V}$

Nonlinearity induced squared **frequency shift** of a single site oscillator

$$\delta_l = \dfrac{3E_l}{2\tilde{\varepsilon}_l} \propto E \qquad (\delta_l = \beta |\psi_l|^2)$$

The relation of the two scales $d_K \leq \Delta_K$ with the nonlinear frequency shift δ_l determines the packet evolution.

Different Dynamical Regimes

Three expected evolution regimes [Flach, Chem. Phys (2010) - S. & Flach, PRE (2010) - Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)]

Δ: width of the frequency spectrum, d: average spacing of interacting modes, δ: nonlinear frequency shift.

Weak Chaos Regime: $\delta < d$, $\quad$ $m_2 \sim t^{1/3}$

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina, PRB (1998) – Pikovsky, & Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: $d < \delta < \Delta$, $\quad$ $m_2 \sim t^{1/2}$ $\longrightarrow$ $m_2 \sim t^{1/3}$

Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

Selftrapping Regime: $\delta > \Delta$

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].

Single site excitations
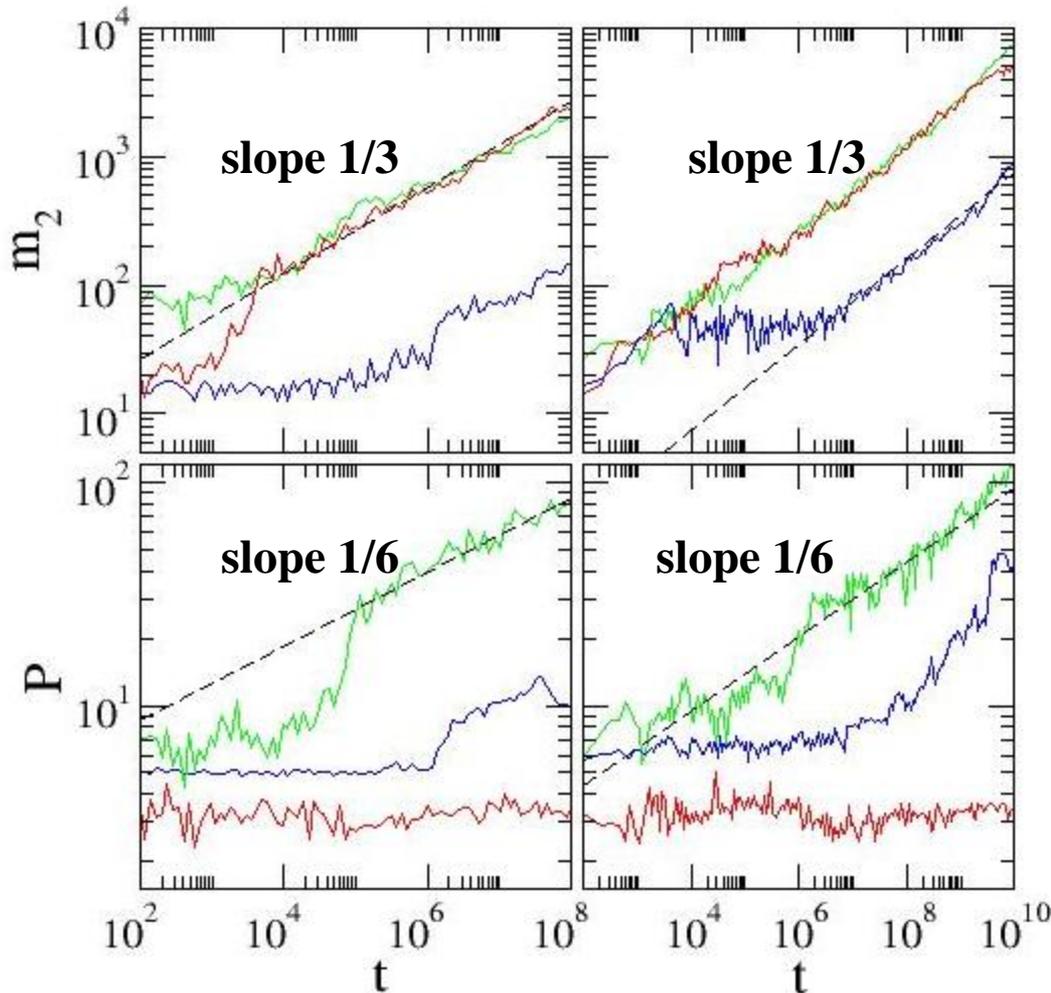
No strong chaos regime

In weak chaos regime we averaged the measured exponent α (m$_2$~t^α) over 20 realizations:

α=0.33±0.05 (KG)
α=0.33±0.02 (DLNS)

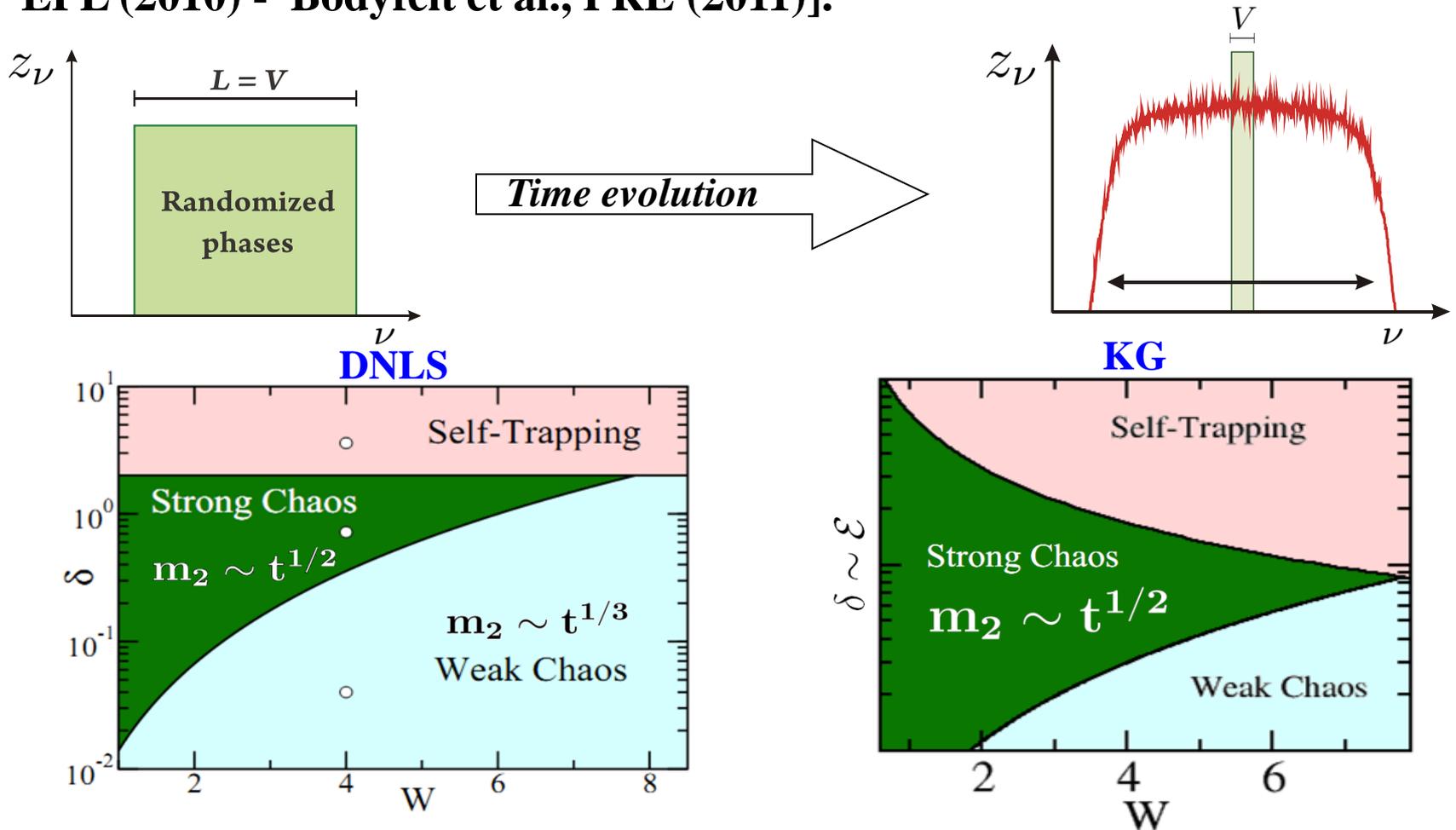**Flach et al., PRL (2009)**
S. et al., PRE (2009)

KG: Different spreading regimes
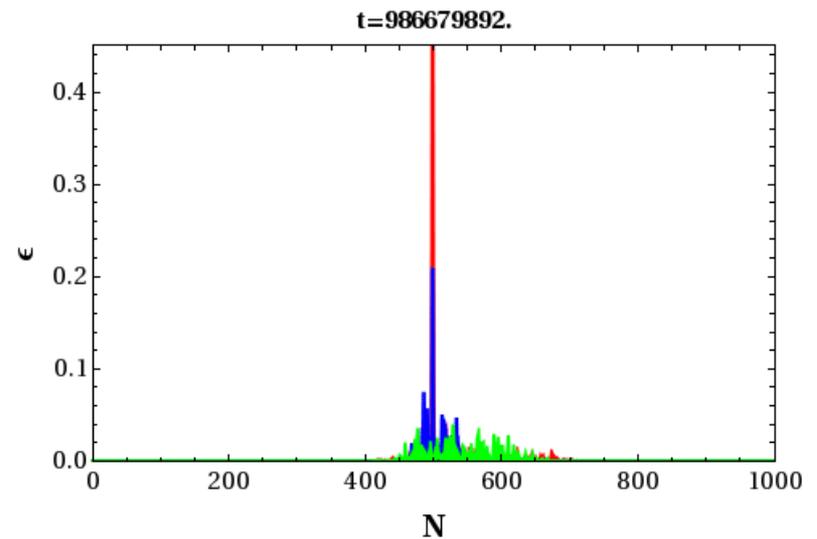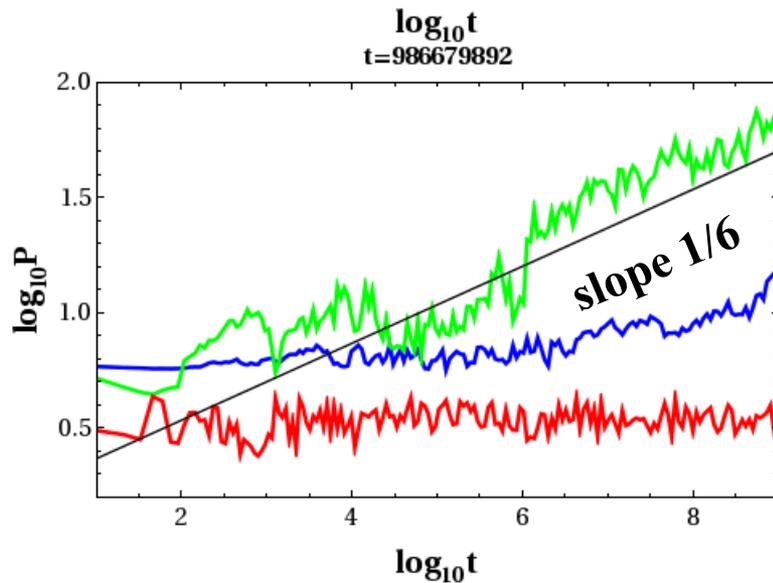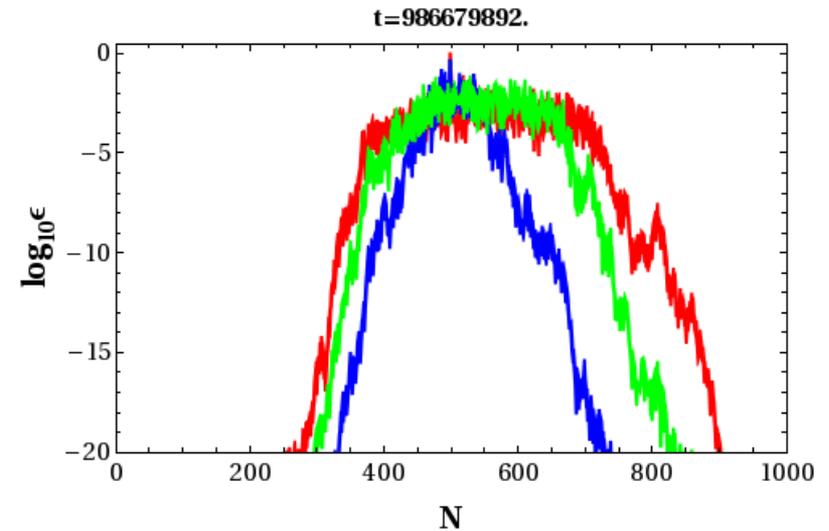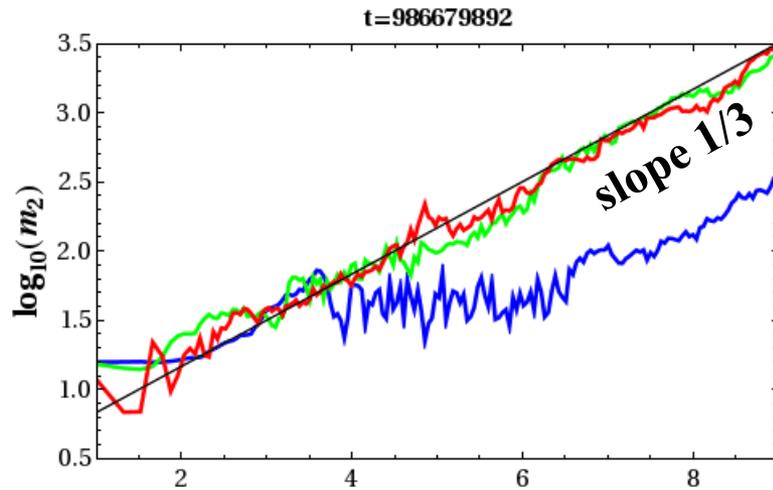
Crossover from strong to weak chaos

We consider **compact initial wave packets of width $L=V$** [Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)].

Crossover from strong to weak chaos (block excitations)

DNLS β= **0.04**, **0.72**, **3.6**   KG E= **0.01**, **0.2**, **0.75**

W=4
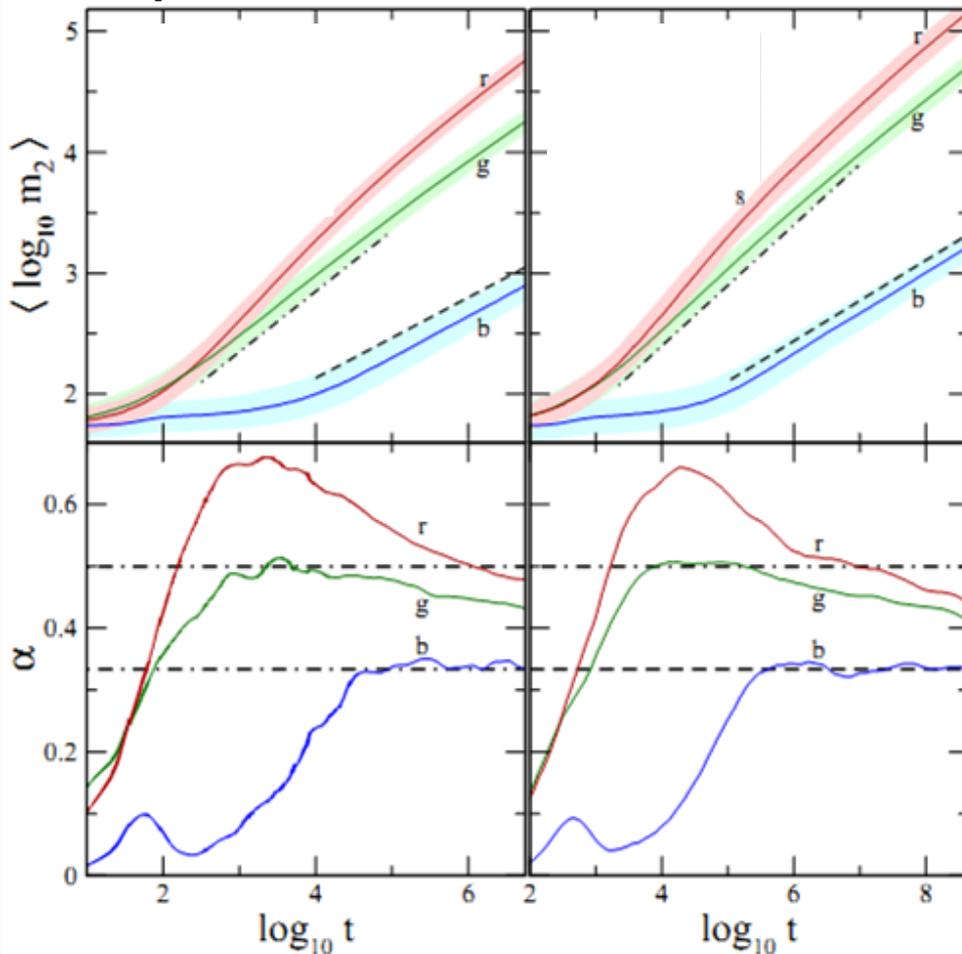
Average over 1000 realizations!

$$\alpha(\log t) = \frac{d\langle \log m_2 \rangle}{d \log t}$$

α=1/2

α=1/3

Laptyeva et al., EPL (2010)

Bodyfelt et al., PRE (2011)

Lyapunov Exponents (LEs)

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:
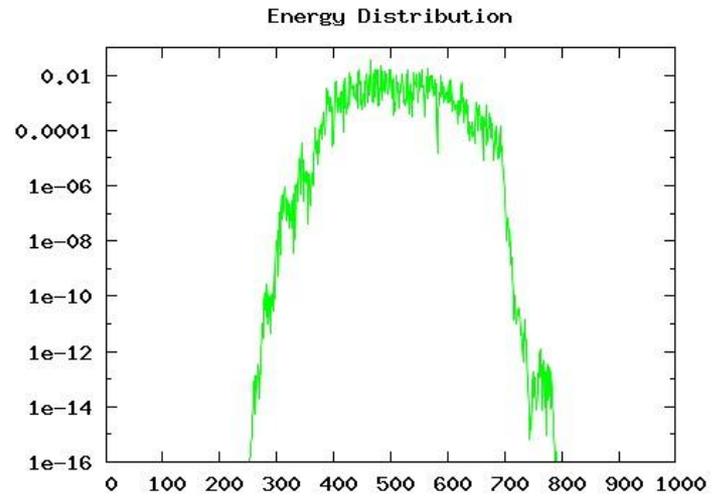
$$\text{mLCE} = \lambda_1 = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\vec{v}(t)\|}{\|\vec{v}(0)\|}$$

$$\lambda_1 = 0 \to \text{Regular motion} \propto (t^{-1})$$
$$\lambda_1 \neq 0 \to \text{Chaotic motion}$$

KG: LEs for single site excitations (E=0.4)
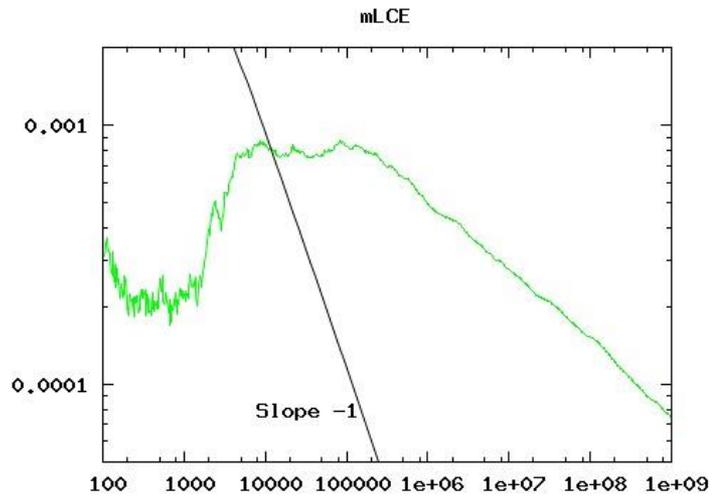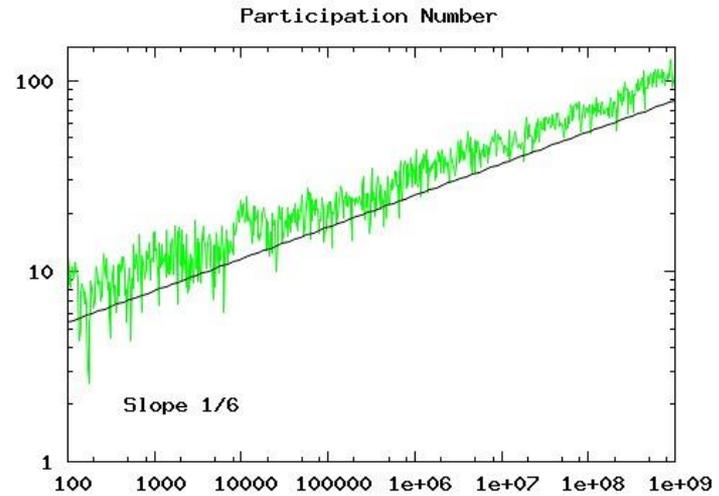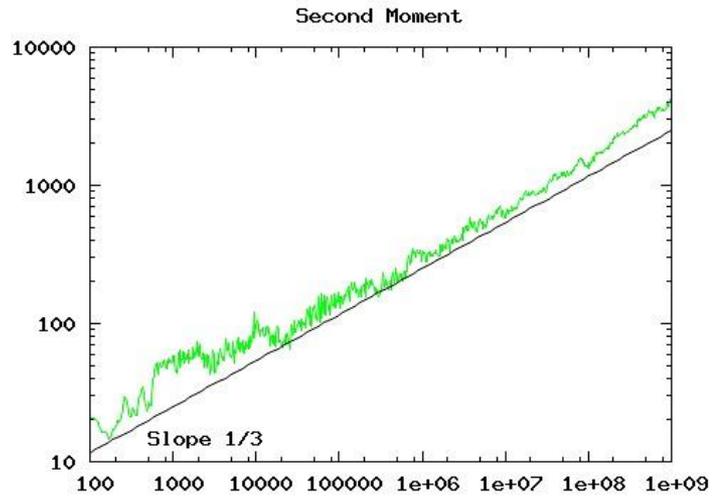
KG: Weak Chaos (E=0.4)

KG: Weak Chaos

Individual runs

Linear case

E=0.4, W=4
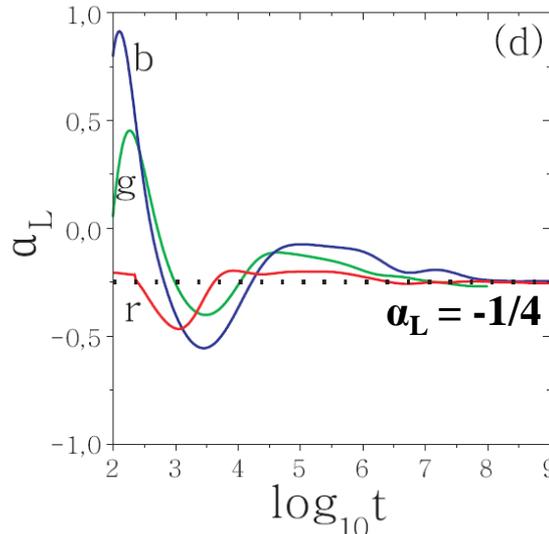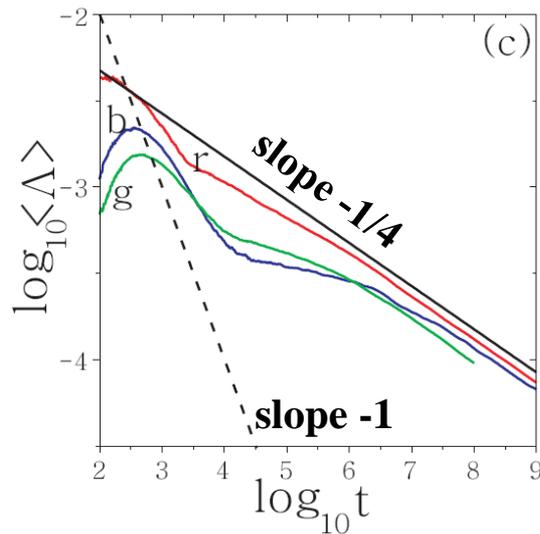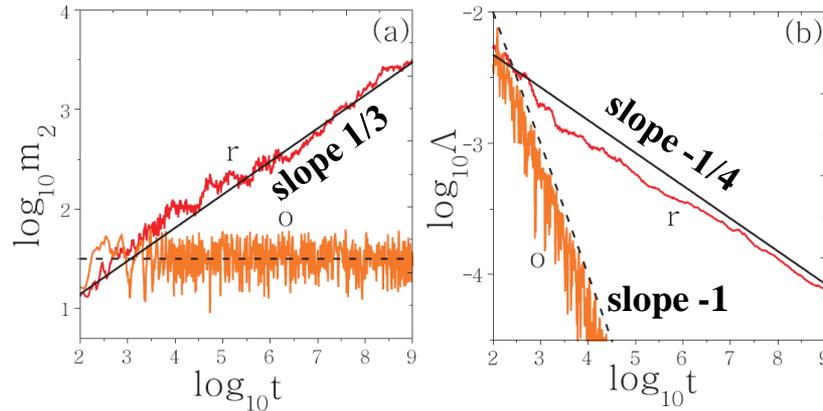
Average over 50 realizations

Single site excitation E=0.4, W=4

Block excitation (21 sites) E=0.21, W=4

Block excitation (37 sites) E=0.37, W=3

$$\alpha_L = \frac{d\left(\log\langle\Lambda\rangle\right)}{d\log t}$$

S. et al. PRL (2013)

Deviation Vector Distributions (DVDs)
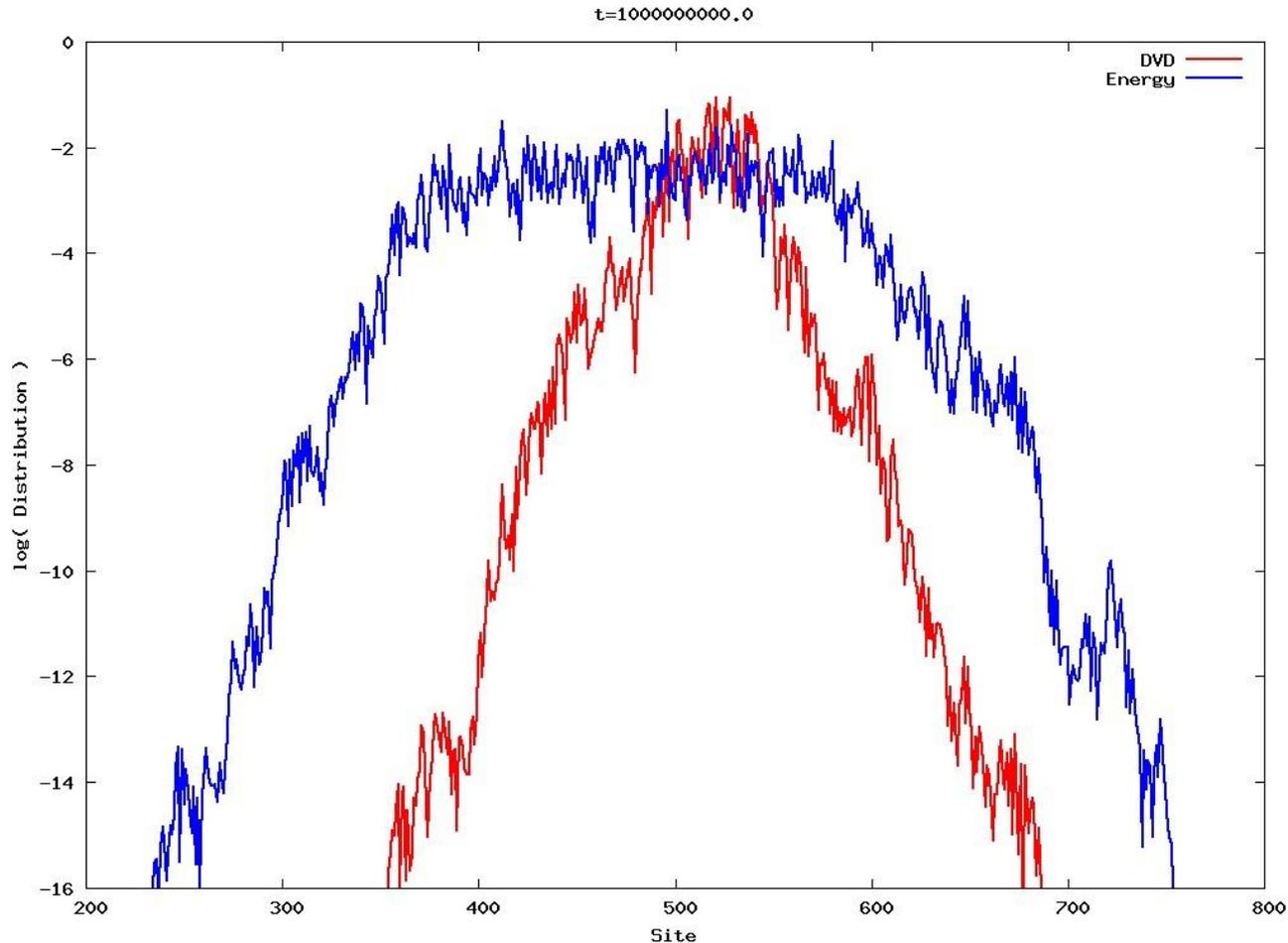
Deviation vector:
$$v(t)=(\delta u_1(t), \delta u_2(t),\dots, \delta u_N(t), \delta p_1(t), \delta p_2(t),\dots, \delta p_N(t))$$

DVD: $w_l = \dfrac{\delta u_l^2 + \delta p_l^2}{\sum\limits_l \left(\delta u_l^2 + \delta p_l^2\right)}$

Deviation Vector Distributions (DVDs)
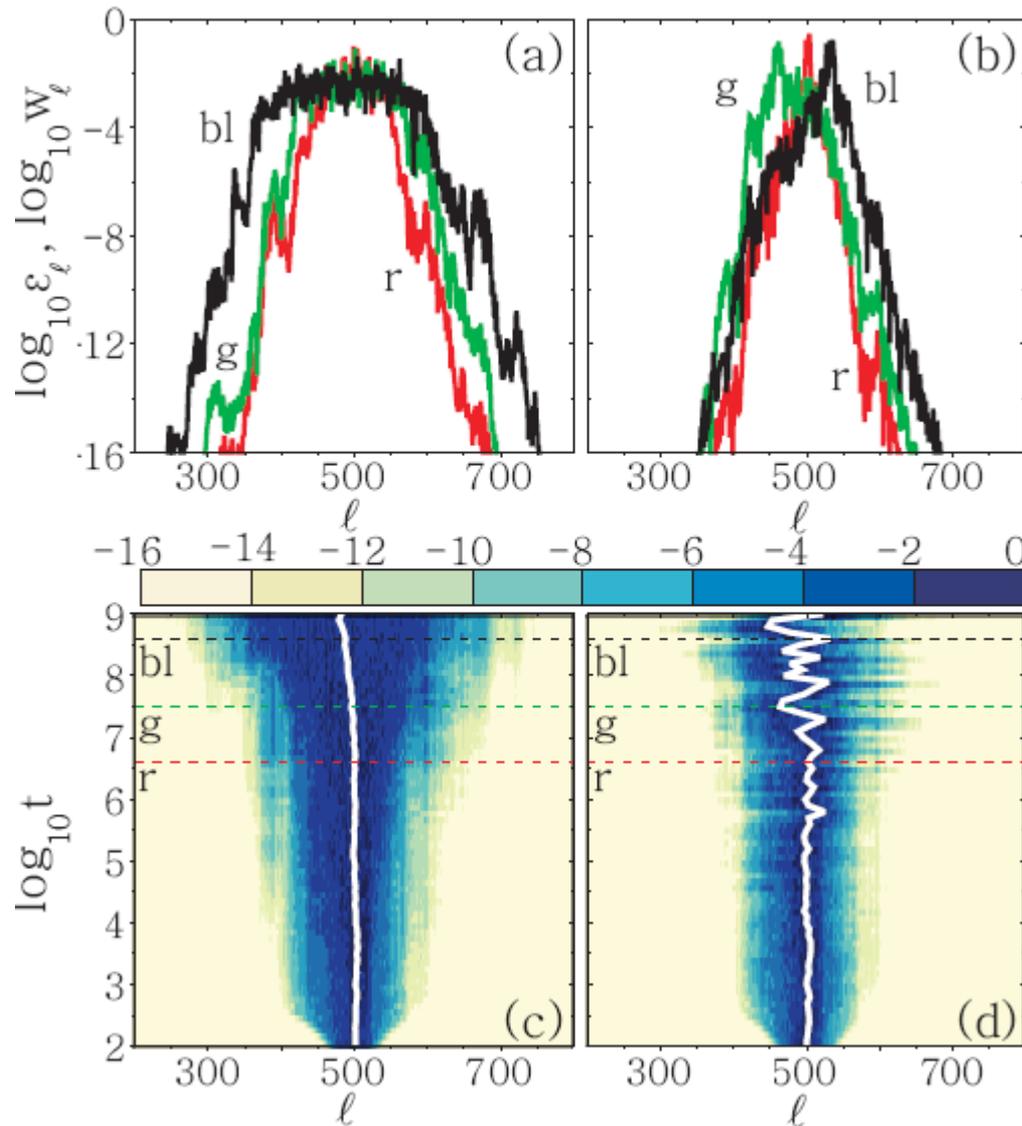
**Individual run
E=0.4, W=4**

Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.

Integration scheme

Consider an N degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

positions   momenta

$$H(q_1, q_2, \ldots, q_N, \ p_1, p_2, \ldots, p_N)$$

The time evolution of an orbit (trajectory) with initial condition

$$P(0) = (q_1(0), \ q_2(0), \ldots, q_N(0), \ p_1(0), \ p_2(0), \ldots, p_N(0))$$

is governed by the Hamilton's equations of motion

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad , \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

Autonomous Hamiltonian systems

Let us consider an **N degree of freedom** autonomous Hamiltonian systems of the form:

$$H(\vec{q}, \vec{p}) = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + V(\vec{q})$$

As an example, we consider the Hénon-Heiles system:

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3} y^3$$

Hamilton equations of motion:

$$\begin{cases} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= y^2 - x^2 - y \end{cases}$$

Variational equations:

$$\begin{cases} \dot{\delta x} &= \delta p_x \\ \dot{\delta y} &= \delta p_y \\ \dot{\delta p}_x &= -(1 + 2y)\delta x - 2x\delta y \\ \dot{\delta p}_y &= -2x\delta x + (-1 + 2y)\delta y \end{cases}$$

Symplectic Integrators (SIs)

Formally the solution of the Hamilton equations of motion can be written as:

$$\frac{d\vec{X}}{dt} = \left\{ H, \vec{X} \right\} = L_H \vec{X} \Rightarrow \vec{X}(t) = \sum_{n \geq 0} \frac{t^n}{n!} L_H^n \vec{X} = e^{tL_H} \vec{X}$$

where $\vec{X}$ is the full coordinate vector and L_H the Poisson operator:

$$L_H f = \sum_{j=1}^{N} \left\{ \frac{\partial H}{\partial p_j} \frac{\partial f}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial f}{\partial p_j} \right\}$$

If the Hamiltonian H can be split into two integrable parts as H=A+B, a symplectic scheme for integrating the equations of motion from time t to time t+τ consists of approximating the operator $e^{\tau L_H}$ by

$$e^{\tau L_H} = e^{\tau(L_A + L_B)} = \prod_{i=1}^{j} e^{c_i \tau L_A} e^{d_i \tau L_B} + O(\tau^{n+1})$$

for appropriate values of constants c_i, d_i. This is an integrator of order n.

So the dynamics over an integration time step τ is described by a series of successive acts of Hamiltonians A and B.

Symplectic Integrator SABA$_2$C

The operator $e^{\tau L_H}$ can be approximated by the symplectic integrator [Laskar & Robutel, Cel. Mech. Dyn. Astr. (2001)]:

$$SABA_2 = e^{c_1 \tau L_A} \; e^{d_1 \tau L_B} \; e^{c_2 \tau L_A} \; e^{d_1 \tau L_B} \; e^{c_1 \tau L_A}$$

with $c_1 = \dfrac{1}{2} - \dfrac{\sqrt{3}}{6}$, $c_2 = \dfrac{\sqrt{3}}{3}$, $d_1 = \dfrac{1}{2}$.

The integrator has only small positive steps and its error is of order 2.

In the case where A is quadratic in the momenta and B depends only on the positions the method can be improved by introducing a corrector C, having a small negative step:

$$C = e^{-\tau^3 \frac{c}{2} L_{\{\{A,B\},B\}}}$$

with $c = \dfrac{2 - \sqrt{3}}{24}$.

Thus the full integrator scheme becomes: $SABAC_2 = C \, (SABA_2) \, C$ and its error is of order 4.

Tangent Map (TM) Method

Use symplectic integration schemes for the whole set of equations (S. & Gerlach, PRE (2010)

We apply the SABAC$_2$ integrator scheme to the Hénon-Heiles system (with ε=1) by using the splitting:

$$A = \frac{1}{2}(p_x^2 + p_y^2), \qquad B = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3,$$

with a corrector term which corresponds to the Hamiltonian function:

$$C = \{\{A, B\}, B\} = (x + 2xy)^2 + (x^2 - y^2 + y)^2$$

We approximate the dynamics by the act of Hamiltonians A, B and C, which correspond to the symplectic maps:

$$e^{\tau L_A} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ p_x' &= p_x \\ p_y' &= p_y \end{cases},$$

$$e^{\tau L_C} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1 + 2x^2 + 6y + 2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \end{cases}$$

$$e^{\tau L_B} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1 + 2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \end{cases},$$

Tangent Map (TM) Method

Let $\vec{u} = (x, y, p_x, p_y, \delta x, \delta y, \delta p_x, \delta p_y)$

The system of the Hamilton's equations of motion and the variational equations is split into two integrable systems which correspond to Hamiltonians A and B.

$$
\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y
\end{aligned}
$$

$$
\begin{aligned}
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= -(1 + 2y)\delta x - 2x\delta y \\
\dot{\delta p}_y &= -2x\delta x + (-1 + 2y)\delta y
\end{aligned}
$$

$A(\vec{p})$ $\longrightarrow$

$$
\left.\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= 0 \\
\dot{p}_y &= 0 \\
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= 0 \\
\dot{\delta p}_y &= 0
\end{aligned}\right\}
\Rightarrow \frac{d\vec{u}}{dt} = L_{AV}\vec{u} \Rightarrow e^{\tau L_{AV}} :
\left\{\begin{aligned}
x' &= x + p_x\tau \\
y' &= y + p_y\tau \\
px' &= p_x \\
py' &= p_y \\
\delta x' &= \delta x + \delta p_x\tau \\
\delta y' &= \delta y + \delta p_y\tau \\
\delta p_x' &= \delta p_x \\
\delta p_y' &= \delta p_y
\end{aligned}\right.
$$

$B(\vec{q})$ $\longrightarrow$

$$
\left.\begin{aligned}
\dot{x} &= 0 \\
\dot{y} &= 0 \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y \\
\dot{\delta x} &= 0 \\
\dot{\delta y} &= 0 \\
\dot{\delta p}_x &= -(1 + 2y)\delta x - 2x\delta y \\
\dot{\delta p}_y &= -2x\delta x + (-1 + 2y)\delta y
\end{aligned}\right\}
\Rightarrow \frac{d\vec{u}}{dt} = L_{BV}\vec{u} \Rightarrow e^{\tau L_{BV}} :
\left\{\begin{aligned}
x' &= x \\
y' &= y \\
p_x' &= p_x - x(1 + 2y)\tau \\
p_y' &= p_y + (y^2 - x^2 - y)\tau \\
\delta x' &= \delta x \\
\delta y' &= \delta y \\
\delta p_x' &= \delta p_x - [(1 + 2y)\delta x + 2x\delta y]\tau \\
\delta p_y' &= \delta p_y + [-2x\delta x + (-1 + 2y)\delta y]\tau
\end{aligned}\right.
$$

Tangent Map (TM) Method

Any symplectic integration scheme used for solving the Hamilton equations of motion, which involves the act of Hamiltonians A and B, can be extended in order to integrate simultaneously the variational equations [S. & Gerlach, PRE (2010) – Gerlach & S., Discr. Cont. Dyn. Sys. (2011) – Gerlach et al., IJBC (2012)].

$$e^{\tau L_A} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ p_x' &= p_x \\ p_y' &= p_y \end{cases}$$

$$e^{\tau L_{AV}} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ px' &= p_x \\ py' &= p_y \\ \delta x' &= \delta x + \delta p_x \tau \\ \delta y' &= \delta y + \delta p_y \tau \\ \delta p_x' &= \delta p_x \\ \delta p_y' &= \delta p_y \end{cases}$$

$$e^{\tau L_B} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1 + 2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \end{cases}$$

$$e^{\tau L_{BV}} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1 + 2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - [(1 + 2y)\delta x + 2x\delta y]\tau \\ \delta p_y' &= \delta p_y + [-2x\delta x + (-1 + 2y)\delta y]\tau \end{cases}$$

$$e^{\tau L_C} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1 + 2x^2 + 6y + 2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \end{cases}$$

$$e^{\tau L_{CV}} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1 + 2x^2 + 6y + 2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - 2\left[(1 + 6x^2 + 2y^2 + 6y)\delta x + 2x(3 + 2y)\delta y\right]\tau \\ \delta p_y' &= \delta p_y - 2\left[2x(3 + 2y)\delta x + (1 + 2x^2 + 6y^2 - 6y)\delta y\right]\tau \end{cases}$$

The KG model

We apply the **SABAC$_2$** integrator scheme to the KG Hamiltonian by using the splitting:

$$H_K = \sum_{l=1}^{N} \left(\frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} \left(u_{l+1} - u_l \right)^2 \right)$$

A **B**

$$e^{\tau L_A}: \begin{cases} u_l' = p_l \tau + u_l \\ p_l' = p_l, \end{cases}$$

$$e^{\tau L_B}: \begin{cases} u_l' = u_l \\ p_l' = \left[-u_l(\tilde{\epsilon}_l + u_l^2) + \frac{1}{W}(u_{l-1} + u_{l+1} - 2u_l) \right]\tau + p_l, \end{cases}$$

with a **corrector term** which corresponds to the Hamiltonian function:

$$C = \left\{ \left\{ A, B \right\}, B \right\} = \sum_{l=1}^{N} \left[u_l(\tilde{\varepsilon}_l + u_l^2) - \frac{1}{W}(u_{l-1} + u_{l+1} - 2u_l) \right]^2 .$$

The DNLS model

A 2nd order SABA Symplectic Integrator with 5 steps, combined with approximate solution for the *B* part (Fourier Transform): SIFT[2]

$$H_D = \sum_l \varepsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l\right), \qquad \psi_l = \frac{1}{\sqrt{2}}\left(q_l + ip_l\right)$$

$$H_D = \sum_l \left(\underbrace{\frac{\varepsilon_l}{2}\left(q_l^2 + p_l^2\right) + \frac{\beta}{8}\left(q_l^2 + p_l^2\right)^2}_{A} \underbrace{- q_n q_{n+1} - p_n p_{n+1}}_{B} \right)$$

$$e^{\tau L_A}: \begin{cases} q_l' = q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\ p_l' = p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau), \end{cases}$$

$$\alpha_l = \epsilon_l + \beta(q_l^2 + p_l^2)/2$$

$$e^{\tau L_B}: \begin{cases} \varphi_q = \sum_{m=1}^{N} \psi_m e^{2\pi i q(m-1)/N} \\ \varphi_q' = \varphi_q e^{2i\cos(2\pi(q-1)/N)\tau} \\ \psi_l' = \frac{1}{N}\sum_{q=1}^{N} \varphi_q' e^{-2\pi i l(q-1)/N} \end{cases}$$

The DNLS model

Symplectic Integrators produced by Successive Splits (SS)

$$H_D = \sum_l \left(\overbrace{\frac{\varepsilon_l}{2}\left(q_l^2 + p_l^2\right) + \frac{\beta}{8}\left(q_l^2 + p_l^2\right)^2}^{A} \underbrace{- q_n q_{n+1} - p_n p_{n+1}}_{B} \right)$$

$$\begin{cases} q_l' = q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\ p_l' = p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau), \end{cases} \begin{cases} q_l' = q_l, \\ p_l' = p_l + (q_{l-1} + q_{l+1})\tau \end{cases} \begin{cases} p_l' = p_l, \\ q_l' = q_l - (p_{l-1} + p_{l+1})\tau \end{cases}$$

Using the SABA$_2$ integrator we get a 2nd order integrator with 13 steps, SS^2:

$$SS^2 = e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right]L_A} e^{\frac{\tau}{2}L_B} e^{\frac{\sqrt{3}\tau}{3}L_A} e^{\frac{\tau}{2}L_B} e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right]L_A}$$

$$\tau' = \tau / 2$$

$$e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right]L_{B_1}} e^{\frac{\tau'}{2}L_{B_2}} e^{\frac{\sqrt{3}\tau'}{3}L_{B_1}} e^{\frac{\tau'}{2}L_{B_2}} e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right]L_{B_1}} e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right]L_{B_1}} e^{\frac{\tau'}{2}L_{B_2}} e^{\frac{\sqrt{3}\tau'}{3}L_{B_1}} e^{\frac{\tau'}{2}L_{B_2}} e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right]L_{B_1}}$$

Three part split symplectic integrators for the DNLS model

Three part split symplectic integrator of order 2, with 5 steps: ABC²

$$H_D = \sum_l \left(\underbrace{\frac{\varepsilon_l}{2}\left(q_l^2 + p_l^2\right) + \frac{\beta}{8}\left(q_l^2 + p_l^2\right)^2}_{A} \underbrace{- q_n q_{n+1}}_{B} \underbrace{- p_n p_{n+1}}_{C} \right)$$

$$ABC^2 = e^{\frac{\tau}{2}L_A} \; e^{\frac{\tau}{2}L_B} \; e^{\tau L_C} \; e^{\frac{\tau}{2}L_B} \; e^{\frac{\tau}{2}L_A}$$

This low order integrator has already been used by e.g. Chambers, MNRAS (1999) – Goździewski et al., MNRAS (2008).

Composition Methods: 4th order SIs

Starting from any 2nd order symplectic integrator S^{2nd}, we can construct a 4th order integrator S^{4th} using the composition method proposed by Yoshida [Phys. Lett. A (1990)]:

$$S^{4th}(\tau) = S^{2nd}(x_1\tau) \times S^{2nd}(x_0\tau) \times S^{2nd}(x_1\tau), \qquad x_0 = -\frac{2^{1/3}}{2 - 2^{1/3}}, \qquad x_1 = \frac{1}{2 - 2^{1/3}}$$

In this way, starting with the 2nd order integrators SS^2, SIFT^2 and ABC^2 we construct the 4th order integrators:

SS^4 with 37 steps   SIFT^4 with 13 steps   ABC$^4_{[Y]}$ with 13 steps

Composition method proposed by Suzuki [Phys. Lett. A (1990)]:

$$S^{4th}(\tau) = S^{2nd}(p_2\tau) \times S^{2nd}(p_2\tau) \times S^{2nd}((1 - 4p_2)\tau) \times S^{2nd}(p_2\tau) \times S^{2nd}(p_2\tau)$$

$$p_2 = \frac{1}{4 - 4^{1/3}}, \qquad 1 - 4p_2 = -\frac{4^{1/3}}{4 - 4^{1/3}}$$

Starting with the 2nd order integrators ABC^2 we construct the 4th order integrator: ABC$^4_{[S]}$ with 21 steps.

More 4ᵗʰ order SIs

We construct few more integration schemes by considering the 4ᵗʰ order symplectic integrators **ABA864, ABA1064, ABAH864 and ABAH1064** introduced by Blanes et al., Appl. Num. Math. (2013) and Farrés et al., Cel. Mech. Dyn. Astr. (2013).

Approximating the solution of the B part by a **Fourier Transform** we construct the 4ᵗʰ order integrators:
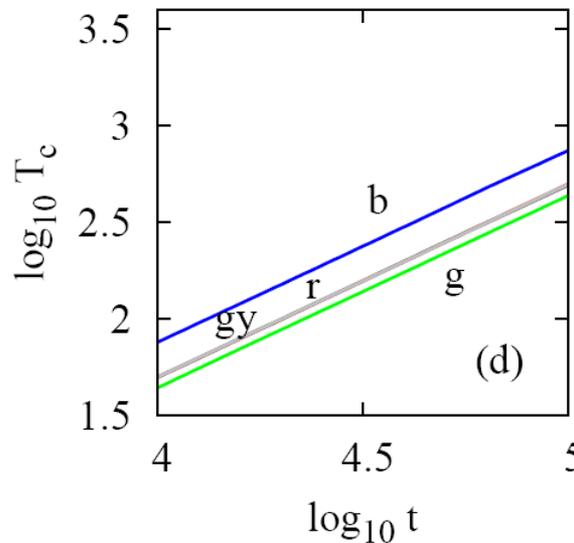
$SIFT^4_{864}$ **with 43 steps** $SIFT^4_{1064}$ **with 49 steps**
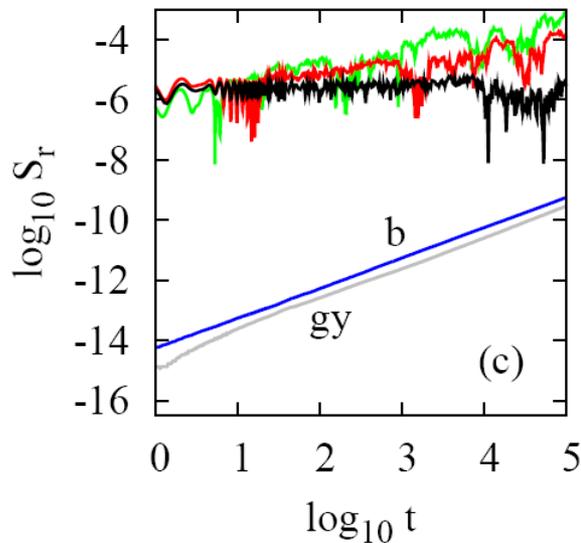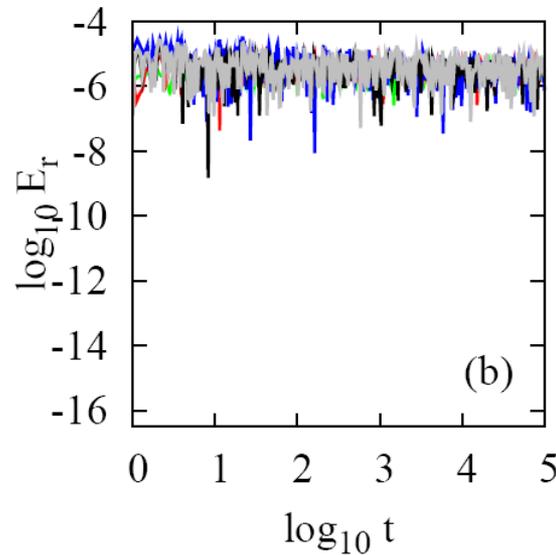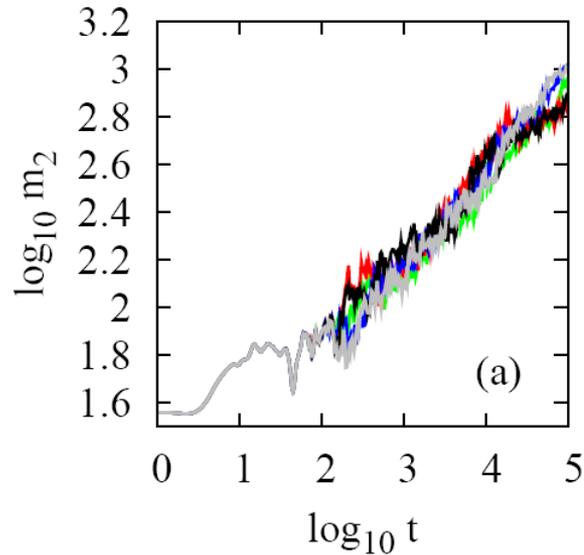
Using **successive splits for the B part** and implementing the $SABA_2$ integrator for its integartion, we construct the 4ᵗʰ order integrators:

SS^4_{864} **with 49 steps** SS^4_{1064} **with 55 steps**

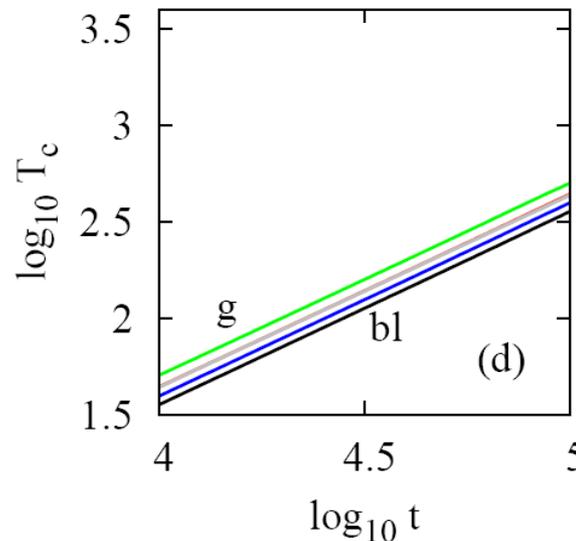# 4th order integrators: Numerical results (I)

SIFT⁴ τ=0.125
SIFT² τ=0.05
ABC⁴[S] τ=0.1
SS⁴ τ=0.1
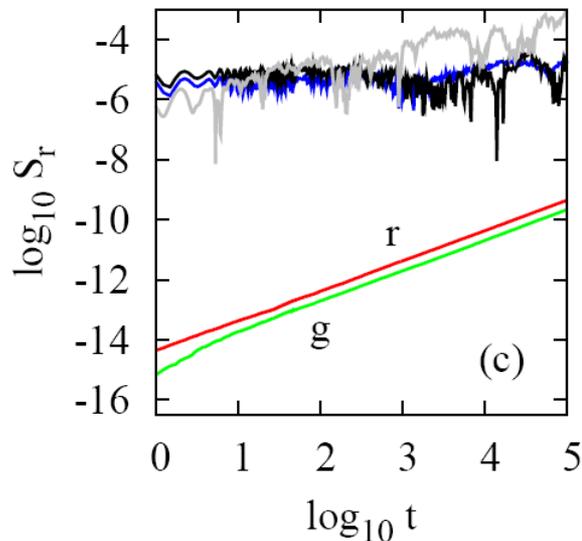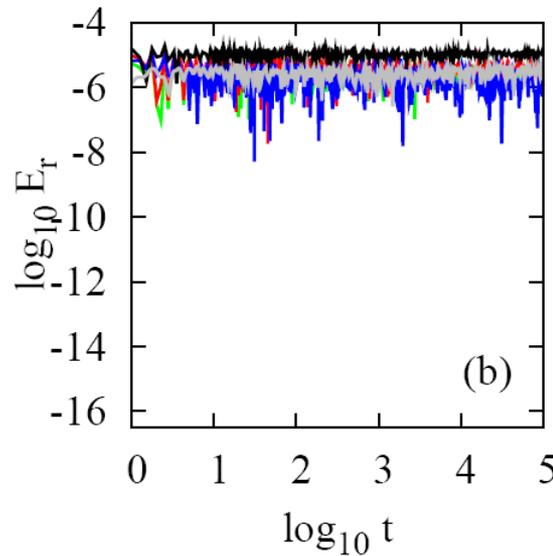ABC⁴[Y] τ=0.05

E_r: relative energy error

S_r: relative norm error

T_c: CPU time (sec)

S. et al., Phys. Lett. A (2014)

4th order integrators: Numerical results (II)
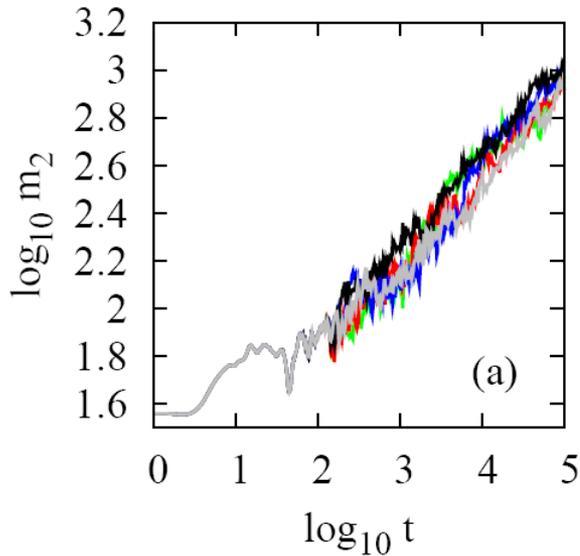
SIFT$^4_{1064}$ τ=0.25
ABC$^4_{[Y]}$ τ=0.05
SIFT$^4_{864}$ τ=0.25
SS$^4_{1064}$ τ=0.25
SS$^4_{864}$ τ=0.25

E_r: relative energy error

S_r: relative norm error

T_c: CPU time (sec)

S. et al., Phys. Lett. A (2014)

Summary (I)

- **We presented three different dynamical behaviors for wave packet spreading in 1d nonlinear disordered lattices:**
  - ✓ **Weak Chaos Regime: $\delta < d$, $m_2 \sim t^{1/3}$**
  - ✓ **Intermediate Strong Chaos Regime: $d < \delta < \Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$**
  - ✓ **Selftrapping Regime: $\delta > \Delta$**

- **Generality of results:**
  - ✓ **Two different models: KD and DNLS,**
  - ✓ **Predictions made for DNLS are verified for both models.**

- **Lyapunov exponent computations show that:**
  - ✓ **Chaos not only exists, but also persists.**
  - ✓ **Slowing down of chaos does not cross over to regular dynamics.**
  - ✓ **Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.**

- **Our results suggest that Anderson localization is eventually destroyed by nonlinearity, since spreading does not show any sign of slowing down.**

Summary (II)

- We presented several **efficient integration methods** suitable for the **integration of the DNLS model,** which are based on **symplectic integration techniques.**

- The construction of symplectic schemes based on **3 part split of the Hamiltonian** was emphasized **(ABC methods).**

- Algorithms based on the integration of the B part of Hamiltonian via **Fourier transforms**, i.e. methods $SIFT^2$, $SIFT^4$, $SIFT^4_{864}$ and $SIFT^4_{1064}$ succeeded in keeping the relative norm error S_r very low. **Drawback:** they require the number of lattice sites to be 2^k, $k \in \mathbb{N}^*$.

- We hope that our results will **initiate future research** both for the theoretical development of new, improved 3 part split integrators, as well as for their applications to different dynamical systems.

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Thank you for your attention